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## Category Theoretic Semantics for Typed Binding Signatures with Recursion

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**Abstract.** We generalise Fiore et al's account of variable binding for untyped cartesian contexts to give an account of binding for either variables or names that may be typed. We do this in an enriched setting, allowing the incorporation of recursion into the analysis. Extending earlier work by us, we axiomatise the notion of context by defining and using the notion of an enriched pseudo-monad  $S$  on  $V\text{-Cat}$ , with leading examples of  $V$  given by  $Set$  and  $\omega Cpo$ , the latter yielding an account of recursion. Fiore et al implicitly used the pseudo-monad  $T_{fp}$  on  $Cat$  for small categories with finite products. Given a set  $A$  of types, our extension to typed binders and enrichment involves generalising from Fiore et al's use of  $[\mathbb{F}, Set]$  to  $[(SA)^{op}, V^A]$ . We define a substitution monoidal structure on  $[(SA)^{op}, V^A]$ , allowing us to give a definition of binding signature at this level of generality, and extend initial algebra semantics to the typed, enriched axiomatic setting. This generalises and axiomatises previous work by Fiore et al and later authors in particular cases. In particular, it includes the Logic of Bunched Implications and variants, infinitary examples, and structures not previously considered such as those generated by finite limits.

**Keywords:** pseudo-monad, pseudo-distributive law, binding, substitution monoidal structure, binding signature

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## 1. Introduction

At MERLIN '03, Power adumbrated an idea for a unified category-theoretic account of variable binding for untyped contexts [25], axiomatising Fiore et al's use of presheaves on  $\mathbb{F}^{op}$  to model untyped cartesian binders [6] and Tanaka's use of presheaves on  $\mathbb{P}^{op}$  to model untyped linear binders [28], where  $\mathbb{F}$ , respectively  $\mathbb{P}$ , is, up to equivalence, the category of finite sets and all functions, respectively all bijections. Since then, the two authors of this paper have checked the details, made some corrections, generalised and developed Power's idea in [30, 32]. In this paper, we extend that work from untyped variable binding to typed binding of both variables and names, and we do it in an enriched setting: the latter allows the incorporation of recursion into our analysis. We also investigate more exotic kinds of contexts such as infinitary contexts [1].

At the same LICS conference at which Fiore et al presented their work, two other approaches to binding were also presented: one by Hofmann [8], and the other by Gabbay and Pitts [7]. The latter work has gradually drawn closer to ours: initially, Gabbay and Pitts expressed their work in terms of Fraenkel-Mostovski set theory, but more recently they have used the Schanuel topos, which is the full subcategory of the presheaf category  $[Inj, Set]$  determined by those functors that preserve pullbacks, where  $Inj$  is, up to equivalence, the category of finite sets and injections [7]. Their work addresses names rather than variables, has led to the development of nominal logic and appropriate tools, and provides a setting in which one can study process calculi such as the  $\pi$ -calculus [3, 21]. So, although we do not deal with their preservation condition in this paper, we develop examples with an eye to names rather than just variables, now optimistic that we will eventually be able to draw the two strands of work together.

Fiore et al replaced the set of terms that one would consider if there was no binding by an object of the category  $[\mathbb{F}, Set]$ , the idea being that the value of a functor  $X$  at  $n$  would denote a set of terms, modulo  $\alpha$ -conversion, containing at most  $n$  variables. They used a monoidal structure on  $[\mathbb{F}, Set]$  to model substitution, and they used the finite product structure of  $[\mathbb{F}, Set]$  to model pairing. They defined a binding signature to consist of a set  $O$  of operations together with a function  $ar : O \rightarrow \mathbb{N}^*$ . Their leading example was given by the untyped  $\lambda$ -calculus

$$M ::= x \mid \lambda x.M \mid MM$$

for which the appropriate binding signature has two operations, one for lambda and one for application, with arities  $\langle 1 \rangle$  and  $\langle 0, 0 \rangle$  respectively:  $\lambda$ -abstraction has one argument and binds one variable, and application has two arguments and binds no variables. They then used the substitution monoidal structure, the finite product structure, and their definition of binding signature to define and characterise initial algebra semantics.

Although they did not explicitly study types, Fiore et al, reasonably but without a conceptual understanding of the situation, regarded the typed setting as a routine generalisation of the untyped one: given a set  $A$  of types, and letting  $\mathbb{F}(A)$  denote the free category with finite coproducts on  $A$ , they had in mind replacing  $[\mathbb{F}, Set]$  by  $[\mathbb{F}(A), Set^A]$ , extending the substitution and pairing structure of  $[\mathbb{F}, Set]$  to  $[\mathbb{F}(A), Set^A]$ , and adding types to the definition of binding signature, confident that their definition and characterisation of initial algebra semantics would extend. A possible arity would consist not only of a finite string of natural numbers  $(n_1, \dots, n_k)$ , but also a total of  $1 + k + \sum_{1 \leq i \leq k} n_i$  types: a type for the codomain of the prospective operation,  $k$  types for its domain, and  $\sum_{1 \leq i \leq k} n_i$  types to give a type for each variable to be bound. The type structure of the simply typed  $\lambda$ -calculus illustrates how this assignment

of types is supposed to work. For instance, the term  $\lambda x.t$  involves three types, one for each of  $\lambda x.t$ ,  $t$  and  $x$ , consistently with the untyped arity for  $\lambda$ -abstraction being  $\langle 1 \rangle$ .

Tanaka did for linear substitution what Fiore et al did for cartesian substitution, again only addressing untyped binding, again with it seeming clear how to extend from the untyped setting to the typed setting [28]. Tanaka's account was essentially the same as Fiore et al's except for the systematic replacement of Fiore et al's use of  $\mathbb{F}^{op}$ , the free category with finite products on 1, by the use of  $\mathbb{P}^{op}$ , the free symmetric monoidal category on 1.

Other authors, notably those of [5] and [17], have considered typed cartesian binders, agreeing with Fiore et al's approach. But they have not addressed linear binders. Nor have they addressed more sophisticated binding structures such as those associated with the Logic of Bunched Implications [27]. And, in particular, their work has not covered the axiomatic framework that we have recently been expounding, which includes a range of examples beyond cartesianness. Here, we shall illustrate our ideas primarily by cartesian binders and with reference to the mixed cartesian and linear binders of the Logic of Bunched Implications [27]. Other possibilities are given by linear binders and by the Logic of Affine Bunched Implications and Linear Logic.

Summarising our earlier work [30, 32], axiomatically, one first chooses a pseudo-monad  $S$  on  $Cat$  to generate contexts. For example, finite product structure, hence the choice of cartesian contexts, corresponds to the pseudo-monad  $T_{fp}$  on  $Cat$ . One then observes that the construction sending a small category  $\mathcal{C}$  to the presheaf category  $[\mathcal{C}^{op}, Set]$ , which we denote by  $\hat{\mathcal{C}}$ , may be characterised as the free colimit completion of  $\mathcal{C}$ . So, except for size, it amounts to giving a second pseudo-monad  $T_{coc}$  on  $Cat$ . The category  $[\mathbb{F}, Set]$  is obtained by applying  $T_{fp}$  to the category 1, then by applying  $T_{coc}$ , yielding  $T_{coc}T_{fp}1$ , and that is typical. To make use of the composite, one requires a pseudo-distributive law of  $T_{fp}$  over  $T_{coc}$ . So we considered arbitrary pseudo-monads  $S$  and  $T$  on  $Cat$  and a pseudo-distributive law  $ST \rightarrow TS$  of  $S$  over  $T$ , yielding a canonical pseudo-monad structure on the composite  $TS$  and hence a canonical substitution monoidal structure on  $TS(1)$ , generalising Fiore et al's substitution monoidal structure on  $[\mathbb{F}, Set]$ . Axiomatising the finite product structure of  $[\mathbb{F}, Set]$  proved straightforward, as, after some delicate thought about the Kleisli construction, did the extension to types: one obtains a substitution monoidal structure on  $[(SA)^{op}, Set^A]$ , duly generalising Fiore et al as required.

This all enriches without fuss except for the substitution monoidal structure: as we explain in Section 4, a substantial difficulty arises at that point. The problem is that the evident generalisation from the 2-category  $Cat$  to the 2-category  $V-Cat$  does not yield a monoidal structure on the  $V$ -category  $TS(1)$  but rather on its underlying ordinary category, which is not sufficient to make the link with substitution. So we need to develop a new enriched category theoretic technique at that point. Ultimately, the new technique required is not conceptually difficult, but it does require attention to detail, involving the notion of an enriched pseudo-monad. In order not to disrupt the flow of the paper, we have placed the required new enriched category theory in Appendix A and, through the course of the paper, we have expressed ourselves without enrichment except for making occasional remarks when the step from the unenriched to the enriched setting involves substantial complication.

Having developed the substitution monoidal structure in [30], we defined a notion of untyped binding signature in [32]. That proved to be quite difficult to do in a way that generalised the earlier definitions, included examples associated with Bunched Implications, and allowed us to define and characterise initial algebra semantics. In extending from the untyped setting to the typed setting, it was that part of the work that required most care. In fact, the effort to extend to types exposed a weakness in our earlier definition of untyped binding signature as explained in [31]: but fortunately, we repaired the weakness

before publication of [32].

For Fiore et al and Tanaka, the number of binders determined an arity, but that is not the case for the Logic of Bunched Implications as one must choose between product or tensor. So for full generality of the notion of arity, one needs an additional parameter in the definition of signature for each natural number of Fiore et al. One also needs to add types systematically. Where Fiore et al have an arity of the form  $(n_1, \dots, n_k)$ , one needs a total of  $1 + k + \sum_{1 \leq i \leq k} n_i$  types.

Fiore et al and Tanaka's central theorems characterised the presheaf of terms generated by a binding signature  $\Sigma$  as the initial  $\Sigma$ -monoid, where a  $\Sigma$ -monoid consisted of a  $\Sigma$ -structure to model application of an element of  $\Sigma$  to a putative term, together with a monoid structure to model substitution, satisfying a natural coherence condition. The central proposition they needed in order to make that characterisation provided a canonical strength for the endofunctor on  $[\mathbb{F}, \text{Set}]$ , respectively  $[\mathbb{P}, \text{Set}]$ , generated by  $\Sigma$ , with respect to the substitution monoidal structure, over any pointed object

$$\Sigma X \bullet Y \longrightarrow \Sigma(X \bullet Y)$$

Here, we prove the same proposition but in a more sophisticated setting to account for examples such as Bunched Implications and types and enrichment. We therefore prove the proposition in regard to the  $V$ -category  $[(SA)^{op}, V^A]$ , and that, with the definitions supporting it, is the heart of the technical content of this paper.

The paper is organised as follows. In Section 2 we describe pseudo-monads on  $Cat$  for variables, locations and names, and one for presheaves, with care for enrichment. In Section 3, we recall the definition of a pseudo-distributive law and discuss its properties, especially in regard to enrichment. In Section 4, we extend our previous analysis of substitution to define and describe a typed substitution monoidal structure generated by a pseudo-monad on  $Cat$ , with concrete descriptions for presheaf categories and with particular care for enrichment. In Section 5, we give an axiomatic definition of a typed binding signature [31]. We construct the requisite strength with respect to the substitution monoidal structure and thereby characterise the presheaf of terms for initial algebra semantics in Section 6. Appendix A develops the new enriched category theory we need to support the technical work in the main part of the paper.

This paper is a journal version developed from our workshop paper [31].

## 2. Pseudo-monads for Variables, Locations and Names, and a Pseudo-monad for Presheaves

The notion of pseudo-monad on  $Cat$  is a variant of the notion of monad on  $Cat$ . There are two ways in which pseudo-monads differ from ordinary monads. First, they must respect natural transformations, i.e., given a natural transformation  $\alpha : H_0 \rightarrow H_1$ , a pseudo-monad  $T$  must provide a natural transformation  $T\alpha : TH_0 \rightarrow TH_1$ , where  $TH_i$  is the functor given by applying the pseudo-monad  $T$  to the arrow  $H_i$  in the category  $Cat$ . Second, the equalities in the axioms for a monad are systematically replaced by coherent isomorphisms. The details of the definition appear in [29, 30].

Just as the notion of pseudo-monad systematically involves the replacement of equalities by coherent isomorphisms in the definition of monad, the notion of pseudo- $T$ -algebra for a pseudo-monad  $T$  is given by systematically weakening the equalities in the definition of algebra to coherent isomorphisms. That duly extends to a 2-category  $Ps\text{-}T\text{-}Alg$ . The details again appear in [29, 30].

For this paper, we need an enriched version of the two constructions, which we give in Appendix A. A fortiori, all 2-monads are pseudo-monads: for 2-monads are precisely those pseudo-monads for which the invertible 2-cells in the definition of pseudo-monad are equalities.

**Example 2.1.** Let  $T_{fp}$  denote the pseudo-monad on  $Cat$  for small categories with finite products. The 2-category  $Ps\text{-}T_{fp}\text{-}Alg$  has objects given by small categories with finite products, maps given by functors that preserve finite products in the usual sense, i.e., up to coherent isomorphism, and 2-cells given by all natural transformations. So  $Ps\text{-}T_{fp}\text{-}Alg$  is the 2-category  $FP$ . The category  $T_{fp}(C)$  is the free category with finite products on  $C$ . Taking  $C = 1$ , the category  $T_{fp}(C)$  is given, up to equivalence, by  $Set_f^{op}$ , which is denoted by  $\mathbb{F}^{op}$  by Fiore et al [6]. More generally, taking  $C$  to be a set  $A$  of types, an object of  $T_{fp}(C)$  would consist of a finite sequence of types, i.e., a context up to  $\alpha$ -equivalence, and a map would amount to exchange, copying, and discarding of variables, respecting types.

Mild variants of Example 2.1 are given by the pseudo-monads  $T_{sm}$  for small symmetric monoidal categories,  $T_{smI}$  for small symmetric monoidal categories whose unit is the terminal object, and  $T_{BI}$  for small symmetric monoidal categories with finite products [30, 32]. Examples that we have not previously considered are as follows:

**Example 2.2.** Following Ambler, Crole and Momigliano [1], Example 2.1 can be modified to allow for countable contexts. Let  $T_{cp}$  denote the pseudo-monad on  $Cat$  for small categories with countable products. The 2-category  $Ps\text{-}T_{cp}\text{-}Alg$  has objects given by small categories with countable products, maps given by functors that preserve countable products in the usual sense, i.e., up to coherent isomorphism, and 2-cells given by all natural transformations. So  $Ps\text{-}T_{cp}\text{-}Alg$  is the 2-category  $CP$ . The category  $T_{cp}(C)$  is the free category with countable products on  $C$ . In particular, the category  $T_{cp}(1)$  is given, up to equivalence, by  $Set_c^{op}$ . More generally, taking  $C$  to be a set  $A$  of types, an object of  $T_{cp}(C)$  consists of a countable sequence of types, i.e., a countable context up to  $\alpha$ -equivalence, and a map is given by countable exchange, copying, and discarding of variables, respecting types.

**Example 2.3.** Consider  $T_{fl}$ , the pseudo-monad on  $Cat$  for small categories with finite limits. This pseudo-monad was studied, in the setting of Section 3 herein, by Cattani and Winskel in their analysis of open maps, with which they studied concurrency [2]. The 2-category  $Ps\text{-}T_{fl}\text{-}Alg$  is  $FL$ , the 2-category of small categories with finite limits and functors that preserve finite limits in the usual sense, with 2-cells given by all natural transformations. Perhaps surprisingly,  $T_{fl}(1)$  is equivalent to  $T_{fp}(1)$ , i.e.,  $Set_f^{op}$ . But one understands it in a somewhat different manner, as all the finite limits, rather than just the finite products, of the category are regarded as an axiomatic part of the structure.

We can refine all the above-mentioned examples by allowing enrichment in a category such as  $\omega Cpo$  in order to incorporate recursion. That is in the spirit of the work of O’Hearn and Tennent on local state [19, 20] and in the ongoing development of that of Gabbay and Pitts [7].

Axiomatically, we generalise from base category  $Cat$  to base category  $V\text{-}Cat$ , where  $V$  is a complete and cocomplete symmetric monoidal closed category, such as  $\omega Cpo$ . Let us denote  $V\text{-}Cat$  by  $W$ . Then  $W$  is a complete and cocomplete symmetric monoidal closed category [13], and we can consider pseudo- $W$ -monads on  $W$ , as defined in Appendix A. For instance, consider the pseudo- $W$ -monad  $T_{fp\omega}$  on  $\omega Cpo\text{-}Cat$  for small  $\omega Cpo$ -enriched categories with finite products. This yields a version of Example 2.1 that allows for recursion through enrichment, cf [22]. The  $W$ -2-category of pseudo-algebras is that of

small  $\omega Cpo$ -enriched categories with finite products, the free such on 1 given by  $\mathbb{F}^{op}$  seen as an  $\omega Cpo$ -enriched category. One can take variants of this too, for instance by allowing for countable cotensors: the free such  $\omega Cpo$ -category on 1 is given by  $\omega Cpo_c^{op}$ , where  $\omega Cpo_c$  is the full subcategory of  $\omega Cpo$  determined by the countably presentable  $\omega$ -cpo's.

Our final example concerns free cocompletions, yielding the pseudo-monad over which the above pseudo-monads are to pseudo-distribute.

**Example 2.4.** For size reasons, there is no interesting pseudo-monad on  $Cat$  for cocomplete categories: small cocomplete categories are necessarily preorders, and the free large cocomplete category on a small category does not lie in  $Cat$ . But there are well-studied techniques to deal with that concern [13, Section 2.6], allowing us safely to ignore it here. Assuming we do that, there is a pseudo-monad  $T_{coc}$  for cocomplete categories. For any small category  $C$ , the category  $T_{coc}(C)$  is given by the presheaf category  $\widehat{C}$ . This construction is fundamental for Fiore et al, Tanaka, O'Hearn and Tennent, Gabbay and Pitts, Pym, and Ambler et al [6, 7, 20, 27, 28, 1]. Its universal property was not considered by them, but, as we established in [30, 32] in the unenriched setting, extended non-trivially to the enriched setting in Section 4 here, it explains why the substitution monoidal structures are definitive. The notion and characterisation of free cocompletion enriches over any complete and cocomplete symmetric monoidal closed category  $V$  and is analysed in detail in [13]: the free cocompletion of a small  $V$ -category  $C$  is the functor- $V$ -category  $[C^{op}, V]$ , which might also reasonably be denoted by  $\widehat{C}$ , as it is the canonical enrichment of the notion of presheaf category.

### 3. Pseudo-Distributive Laws and Strict Liftings

In this section, we first recall the definition of pseudo-distributive law and the fact that to give a pseudo-distributive law is equivalent to giving a strict lifting. In examples, the latter concept is typically more primitive (see below and also Cattani and Winskel's work on profunctors and open maps [2]), but formally, it is more elegant to take the notion of pseudo-distributive as primitive, then immediately prove a characterisation theorem.

If one systematically generalises from two-dimensional structure to  $W$ -enriched structure for a symmetric monoidal closed 2-category as detailed in Appendix A, the equivalence and all further abstract theory of the section remain true. With one exception, all our examples also enrich directly. We explain the exception, that of finite limits, in detail: it does enrich, but only with further conditions on  $V$  and only with a generalisation from finiteness to countable presentability. In general, we continue to express ourselves primarily in terms of unenriched structure in this section for ease of exposition.

An ordinary distributive law consists of a pair of monads  $S$  and  $T$  and a natural transformation  $\delta : ST \rightarrow TS$ , satisfying four coherence conditions. The coherence axioms for an ordinary distributive law correspond exactly to the data for a pseudo-distributive law [29].

**Definition 3.1.** Given a 2-category  $\mathbb{C}$  and pseudo-monads  $(S, \mu^S, \eta^S, \tau^S, \lambda^S, \rho^S)$  and  $(T, \mu^T, \eta^T, \tau^T, \lambda^T, \rho^T)$  on  $\mathbb{C}$ , a *pseudo-distributive law*  $(\delta, \bar{\mu}^S, \bar{\mu}^T, \bar{\eta}^S, \bar{\eta}^T)$  of  $S$  over  $T$  consists of

- a pseudo-natural transformation  $\delta : ST \rightarrow TS$

- invertible modifications

$$\begin{array}{ccccc}
S^2T & \xrightarrow{S\delta} & STS & \xrightarrow{\delta S} & TS^2 \\
\downarrow \mu^S T & & \Downarrow \bar{\mu}^S & & \downarrow T\mu^S \\
ST & \xrightarrow{\delta} & TS & & TS
\end{array}$$
  

$$\begin{array}{ccccc}
ST^2 & \xrightarrow{\delta T} & STS & \xrightarrow{T\delta} & T^2S \\
\downarrow S\mu^T & & \Downarrow \bar{\mu}^T & & \downarrow \mu^T S \\
ST & \xrightarrow{\delta} & TS & & TS
\end{array}$$
  

$$\begin{array}{ccc}
T & & S \\
\downarrow \eta^S T & \nearrow T\eta^S & \downarrow S\eta^T \\
ST & \xrightarrow{\delta} & TS
\end{array}$$

subject to ten coherence axioms [30, 29].

By a *lifting* of a pseudo-monad  $T$  to the 2-category  $Ps\text{-}S\text{-}Alg$  of pseudo-algebras for a pseudo-monad  $S$ , we mean a pseudo-monad  $\tilde{T}$  on the 2-category  $Ps\text{-}S\text{-}Alg$  such that  $U_S \tilde{T} = T U_S$ , and similarly for the other data, where  $U_S$  is the forgetful 2-functor for the pseudo-monad  $S$ .

**Theorem 3.1. ([29, 30])**

To give a pseudo-distributive law  $\delta : ST \longrightarrow TS$  of pseudo-monads on  $Cat$  is equivalent to giving a lifting of the pseudo-monad  $T$  to a pseudo-monad  $\tilde{T}$  on  $Ps\text{-}S\text{-}Alg$ .

**Theorem 3.2. ([29, 30])**

Given a pseudo-distributive law  $\delta : ST \longrightarrow TS$  of pseudo-monads on  $Cat$

- the pseudo-functor  $TS$  acquires the structure for a pseudo-monad, with multiplication given by

$$TSTS \xrightarrow{T\delta S} TTSS \xrightarrow{\mu^T \mu^S} TS$$

- $Ps\text{-}TS\text{-}Alg$  is canonically isomorphic to  $Ps\text{-}\tilde{T}\text{-}Alg$
- for every small category  $C$ , the category  $TS(C)$  has both canonical pseudo- $S$ -algebra and pseudo- $T$ -algebra structures on it.

The combination of Theorems 3.1 and 3.2 yields a selection of pseudo-monads on  $Cat$  by combining most of the earlier examples with the last in Section 2. The central result that makes all the examples work is as follows.



**Theorem 3.3. ([30])**

The pseudo-monad for free cocompletions lifts from  $Cat$  to  $SymMon_{str}$ .

**Example 3.1.** Theorem 3.2 restricts to small categories with finite products. Applying Theorems 3.1, 3.2 and 3.3 to  $T_{fp}$  and  $T_{coc}$ , one obtains the pseudo-monad  $T_{coc}T_{fp}$  with  $T_{coc}T_{fp}(1)$  being equivalent to  $[\mathbb{F}, Set]$ , which was Fiore et al's category for cartesian variable binding [6]. Further details of this example seen in the light of pseudo-distributive laws appear in [30].

Similarly, one obtains pseudo-monad structures on  $T_{coc}T_{sm}$ ,  $T_{coc}T_{smI}$ , and  $T_{coc}T_{BI}$  [30, 32]. We do not have a general result characterising those pseudo-monads  $S$  that support a pseudo-distributive law over  $T_{coc}$ . The situation is delicate as the pseudo-monads for finite products and monoidal structure are positive examples while that for finite coproducts is a non-example. It is true of operads and of all examples we can imagine of natural context-forming operations. The analysis can be extended from finite products to countable products as in Example 2.2 and it enriches without fuss. For finite limits, one needs a variant of Theorem 3.3 that is implicit in Cattani and Winskel's work on open maps [2]:

**Theorem 3.4.** The pseudo-monad for free cocompletions lifts from  $Cat$  to  $FL$ .

**Proof:**

For any small category  $C$ , the Yoneda embedding  $\mathcal{Y} : C \longrightarrow \widehat{C}$  preserves finite limits, in fact any limits that exist in  $C$ . The Yoneda embedding is the unit  $\eta_C : C \longrightarrow T_{coc}C$  of the pseudo-monad  $T_{coc}$ . So it remains to check that the multiplication of  $T_{coc}$  when applied to  $C$  preserves finite limits.

For any small category  $C$  with finite limits,  $T_{coc}C = \widehat{C}$  is the free cocompletion of  $C$  that respects the finite limits of  $C$ : if a functor  $H : C \longrightarrow D$  preserves finite limits, and  $D$  is cocomplete and has all finite limits, it follows that the left Kan extension of  $H$  along  $\mathcal{Y}$  also preserves finite limits [14]. Using this fact, one can mimic the latter half of the proof of Theorem 3.3 to deduce that  $\mu_C$  must preserve finite limits.  $\square$

This yields the following example, extending Example 2.3:

**Example 3.2.** Applying Theorems 3.1, 3.2 and 3.4 to  $T_{fl}$  and  $T_{coc}$ , one obtains the pseudo-monad  $T_{coc}T_{fl}$  with  $T_{coc}T_{fl}(1)$  equivalent to  $[\mathbb{F}, Set]$ , but here equipped with finite limit structure rather than just finite product structure.

In contrast to the previous examples, in order to enrich this example, one needs additional conditions on  $V$ . In order to define a sensible notion of finite limit, one typically takes  $V$  to be locally finitely presentable as a symmetric monoidal closed category [14]. The category  $\omega Cpo$  is not locally finitely presentable, but it is locally countably presentable as a symmetric monoidal closed category. So one can enrich Example 3.2 in  $\omega Cpo$  if one is willing to generalise from finite limits to countably presentable limits.

## 4. Substitution

**Definition 4.1.** For any pseudo-monad  $T$  on a 2-category  $\mathbb{C}$ , the *Kleisli bicategory*, denoted  $Kl(T)$ , is the bicategory defined as follows:

- $ob(Kl(T)) = ob(\mathbb{C})$
- $Kl(T)(C, D) = \mathbb{C}(C, TD)$
- composition is determined by the action of  $T$  on homs, the composition structure of  $\mathbb{C}$ , and the multiplication of  $T$ , yielding a canonical map of the form

$$\mathbb{C}(D, TE) \times \mathbb{C}(C, TD) \longrightarrow \mathbb{C}(C, TE)$$

with the evident labelling

with the rest of the bicategory structure determined canonically by the pseudo-monad structure of  $T$ .

Since  $Kl(T)$  is a bicategory, for every object  $C$  of it, the composition of  $Kl(T)$  determines a monoidal structure on the category  $Kl(T)(C, C)$ , i.e., on  $\mathbb{C}(C, TC)$ . If  $\mathbb{C}$  was  $Cat$ , we could then take  $C$  to be 1, yielding a monoidal structure on the category  $Cat(1, T1)$ , which is isomorphic to  $T1$ . Taking a dual, i.e., using the bicategorical structure of  $Kl(T)^{op}$ , we recover the substitution monoidal structure of [6, 17, 28], as shown in [30].

But that argument only works for  $\mathbb{C} = Cat$ . If  $\mathbb{C}$  was  $V-Cat$ , it would yield a monoidal structure on  $V-Cat_0(\mathcal{I}, T\mathcal{I})$ , where  $\mathcal{I}$  is the unit  $V$ -category (see Appendix A), which is not the  $V$ -category  $T\mathcal{I}$  but rather its underlying ordinary category. For instance, taking  $V$  to be  $\omega Cpo$ , one would obtain a monoidal structure on the underlying ordinary category of  $T\mathcal{I}$  rather than on  $T\mathcal{I}$  qua  $\omega Cpo$ -enriched category.

A priori, that might not seem to be a problem. But it is a problem because, in practice, we need to be able to calculate with the monoidal structure in terms of a formula that describes its action on a pair of objects of  $T\mathcal{I}$ . Theorem 4.1 will provide such a formula, but our proof of Theorem 4.1 crucially uses the fact of every presheaf being a colimit of representables, which, in the enriched setting, is only true when one regards the presheaf category as an enriched category [13]. For instance, for the proof of Theorem 4.1 to work for enrichment in  $\omega Cpo$ , one needs to consider  $T\mathcal{I} = [(S1)^{op}, \omega Cpo]$  as an  $\omega Cpo$ -category. Theorem 4.1 is central to our analysis as it asserts that the canonical monoidal structure we have found is inherently a category theoretic formulation of substitution. Thus, for the enriched setting, we need to find a way in which to make  $Kl(T)(C, D)$  into a  $V$ -category rather than a mere category.

Appendix A resolves the situation for us: if we take  $\mathbb{C}$  to be a  $W$ -category for a symmetric monoidal closed 2-category  $W$ , and we take  $T$  to be a pseudo- $W$ -monad on  $\mathbb{C}$ , it follows that  $Kl(T)(C, D)$  is an object of  $W$ . Taking  $W$  to be  $V-Cat$ , and letting  $C$  be any small  $V$ -category, we thus have a monoidal structure  $\bullet$  on  $V-Cat(C, TC)$  qua  $V$ -category. So in particular, taking  $C$  to be  $\mathcal{I}$ , we have a canonical monoidal structure on the  $V$ -category  $T\mathcal{I}$ . Using Appendix A, we can characterise  $\bullet$ .

Given an arbitrary pseudo-monad  $S$  on  $Cat$ , let  $(C, a)$  be (part of) an arbitrary pseudo- $S$ -algebra, and let  $\alpha$  be an object of the category  $Sk$  for any small category  $k$ , in particular for any natural number. The object  $\alpha$  induces a functor  $\bar{\alpha}_C : C^k \rightarrow C$  as follows:

$$C^k \cong C^k \times 1 \xrightarrow{S \times \alpha} (SC)^{Sk} \times Sk \xrightarrow{ev_\alpha} SC \xrightarrow{a} C$$

This enriches directly using Appendix A. In terms of the enrichment, we can characterise  $\bullet$  as follows:

**Theorem 4.1.** Let  $W$  be  $V\text{-Cat}$ . Then, given a pseudo- $W$ -monad  $S$  on  $W$  and a pseudo-distributive law of  $S$  over  $T_{coc}$ , and given  $X, Y$  in  $[(SA)^{op}, V^A]$ , one can calculate the value of  $Xa \bullet_A Y$  at  $c \in SA$  as

$$(Xa \bullet_A Y)_c = \int^{c' \in SA} (Xa)c' \otimes (\overline{c'}_{\widehat{SA}}(Y))c \quad (1)$$

**Proof:**

It follows from the enriched Yoneda lemma [13] that  $Xa$  is the colimit of representables

$$(Xa)- = \int^{c' \in SA} (Xa)c' \otimes (SA)(-, c') \quad (2)$$

But  $c' = \overline{c'}_{SA}(\iota_A)$ , where  $\iota$  is the unit of  $S$ . So, cf [30], we have

$$(Xa)- = \int^{c' \in SA} (Xa)c' \otimes \overline{c'}_{\widehat{SA}}(\mathcal{Y}\iota_A) \quad (3)$$

But  $-\bullet_A Y$  is a map of pseudo- $T_{coc}S$ -algebras, hence a map of pseudo- $T_{coc}$ -algebras and a map of pseudo- $S$ -algebras by the enriched version of Theorem 3.2. So  $-\bullet_A Y$  respects both  $\overline{(-)}$  and all colimits. Moreover,  $\mathcal{Y}\iota$  is the unit  $\eta$  of the monoidal structure defined by  $\bullet$ . So, replacing each occurrence of  $\mathcal{Y}\iota_A$  in Equation (3) by  $Y$ , we have the result.  $\square$

**Example 4.1.** Let  $V = \omega Cpo$  and  $S = T_{fp}$ , with a set  $A$  of types. So the tensor product of  $V$  is finite product. Applying Theorem 4.1, the  $a$  component is calculated as

$$(Xa \bullet_A Y)_c = \int^{c' \in T_{fp}A} (Xa)c' \times (\overline{c'}_{\widehat{T_{fp}A}}(Y))c$$

where  $c, c'$  are elements of  $T_{fp}A$ , which means they are products of elements of  $A$ . The functor  $\overline{c'}_{\widehat{T_{fp}A}}$  is of type

$$\overline{c'}_{\widehat{T_{fp}A}} : [(T_{fp}A)^{op}, \omega Cpo]^A \rightarrow [(T_{fp}A)^{op}, \omega Cpo]$$

Note that  $Y$  is in  $[(T_{fp}A)^{op}, \omega Cpo]^A$ , which is the domain of  $\overline{c'}_{\widehat{T_{fp}A}}$ . If  $c'$  is of the form  $a_1 \times \dots \times a_n$ , one can calculate  $\overline{c'}_{\widehat{T_{fp}A}}(Y)$  as

$$Ya_1 \times \dots \times Ya_n,$$

which we denote here by  $Y^{(c')}$ . Using this one has

$$(Xa \bullet_A Y)_c = \int^{c' \in T_{fp}A} (Xa)c' \times (Y^{(c')})c$$

Since a coend can be calculated as a coequaliser:

$$(Xa \bullet_A Y)_c = ( \coprod_{c' \in T_{fp}A} (Xa)c' \times (Y^{(c')})c ) / \sim$$

where the relation  $\sim$  is defined as follows: letting  $c' = a_1 \times \dots \times a_n$  and  $c'' = a'_1 \times \dots \times a'_{n'}$  be elements of  $T_{fp}A$ , we consider two elements in the above coproduct  $(u; v_1, \dots, v_n)$  and  $(u'; v'_1, \dots, v'_{n'})$ , where

$u \in (Xa)c'$ ,  $u' \in (Xa)c''$ ,  $v_i \in (Ya_i)c$ , for  $1 \leq i \leq n$ , and  $v'_j \in (Ya'_j)c$ , for  $1 \leq j \leq n'$ . Then the two elements are equivalent

$$(u; v_1, \dots, v_n) \sim (u'; v'_1, \dots, v'_{n'})$$

if there exists an arrow  $f : c' \rightarrow c''$  in  $(T_{fp}A)^{op}$  such that  $Xa(f)(u) = u'$ , and for all  $i \in \{1, \dots, n\}$ ,  $f(a_i) = a'_j$  and  $v_i = v'_j$ . An unenriched version of this example essentially corresponds to the case  $T = U$  in [17, Section 3.3], where the semantic category  $\mathcal{C}$  is set to be the syntactic model  $\mathcal{S}$ .

## 5. Typed Binding Signatures

In this section, we define and start to develop the notion of a typed binding signature. We express ourselves in an unenriched setting for ease of exposition. At the end of the section, we explain what modifications need to be made in order to enrich the work.

Fiore et al and Tanaka each defined a binding signature to consist of a set of operations  $O$  together with an arity function  $a : O \rightarrow \mathbb{N}^*$ . Supposing for simplicity they had just one operation with arity  $(n_i)_{1 \leq i \leq k}$ , Fiore et al generated the endofunctor on  $[\mathbb{F}, Set]$  sending  $X$  to

$$(\delta^{n_1} X) \times \dots \times (\delta^{n_k} X)$$

where  $\delta X$  was defined to be  $X(1 + -)$ , giving a mathematical formulation of the idea of binding over one variable. The composite  $\delta^n X$ , which was therefore  $X(n + -)$ , allowed the formulation of the idea of binding over  $n$  variables. But that is not subtle enough in more complex binding situations such as that of Bunched Implications, which has two sorts of binders: a linear binder and a non-linear binder.

The definition of untyped binding signature by Fiore et al essentially contains two pieces of data: for each  $i$ , each  $n_i$  tells you how many times to apply  $X(1 + -)$ , and  $k$  tells you how many such  $X(n_i + -)$  need to be multiplied. But in more complex settings, we need more specificity as a finite sequence of natural numbers does not specify which sort of binder is to be used, and in what combination are the binders to be used: Fiore et al used cartesian binders and took a product; Tanaka used linear binders and took a tensor product; but in Bunched Implications, one has a choice of binders and a choice of product or tensor. To add types, we need to ascribe types both at every occurrence of a potential variable and at every term produced by binding.

There are several equivalent ways in which one can formulate the definition. We formulate it in a way that coheres with definitions in specific cases [10, 5, 17], where each arity is associated to a set of operators, allowing overloading of operator symbols.

**Definition 5.1.** Given a pseudo-monad  $S$  on  $Cat$ , a *typed binding signature*  $\Sigma = (A, \mathcal{O})$  consists of a set of types  $A$ , together with an arity function  $\mathcal{O} : Ar_{S,A} \rightarrow Set$ , where an element  $(k, \alpha, (n_i, \beta_i)_{1 \leq i \leq k}, type)$  of  $Ar_{S,A}$  is given by a natural number  $k$ , an object  $\alpha$  of the category  $Sk$ ,  $k$  pairs of a natural number  $n_i$  and an object  $\beta_i$  of the category  $S(n_i + 1)$ , for  $1 \leq i \leq k$ , and a list *type* of  $1 + k + \sum_{1 \leq i \leq k} n_i$  types, i.e., elements of  $A$ .

The *type* is a sequence of elements of  $A$  of the form

$$\langle a^{out}, \vec{a}^{in}, \vec{a}_1^{bind}, \dots, \vec{a}_k^{bind} \rangle$$

where  $\vec{a}^{in} = a_1^{in}, \dots, a_k^{in}$  and  $\vec{a}_i^{bind} = a_{i,1}^{bind}, \dots, a_{i,n_i}^{bind}$ , with  $a^{out}$  representing the type of codomain of the operator,  $a_i^{in}$  that of domain of the  $i$ th argument,  $1 \leq i \leq k$ , and  $a_{i,j}^{bind}$  that of the  $j$ th variable to be bound in the  $i$ th argument,  $1 \leq i \leq k$  and  $1 \leq j \leq n_i$ , respectively. The functor  $\overline{\beta}_{iS1}(1, \dots, 1, -)$  generalises Fiore et al's  $n_i + -$ .

For a definition of untyped binding signature, one can of course simply strip the types in the definition of typed binding signature. It is routine to check that the result of doing that is a reorganisation of the following definition, which appeared in [32], which in turn improved upon earlier workshop papers [31]:

**Definition 5.2.** For a pseudo-monad  $S$  on  $Cat$ , an *untyped binding signature*  $\Sigma = (O, a)$  is a set of operations  $O$  together with an arity function  $a : O \rightarrow Ar_S$  where an element  $(k, \alpha, (n_i, \beta_i)_{1 \leq i \leq k})$  of  $Ar_S$  consists of a natural number  $k$  and an object  $\alpha$  of the category  $Sk$ , together with, for  $1 \leq i \leq k$ , a natural number  $n_i$  and an object  $\beta_i$  of the category  $S(n_i + 1)$ .

With the definition of typed binding signature in hand, we can induce a signature endofunctor, as Fiore et al and Tanaka did, then speak of the algebras for the endofunctor. For typographical reasons, we need some notational abbreviations as follows. In all interesting cases, the unit  $\iota$  of  $S$  yields, for any set  $A$ , an injective function  $\iota_A : A \rightarrow SA$ . So we shall identify an element  $a$  of  $A$  with its image under  $\iota_A$ . When we write  $\vec{a}$ , we shall often mean a list of types  $a_1, \dots, a_k$ , and when we write  $\vec{b}_i$ , we shall often mean, for each  $1 \leq i \leq k$ , a list of types  $b_{i,1}, \dots, b_{i,n_i}$ . We further abbreviate an expression of the form  $f(x_1, \dots, x_n)$  to  $f(\vec{x})$  and one of the form  $g(y_1, \dots, y_n, -)$  to  $g(\vec{y}, -)$ . Using these notational abbreviations, we define the induced signature endofunctor as follows:

**Proposition 5.1.** Each typed binding signature  $\Sigma$  induces an endofunctor on  $[(SA)^{op}, Set^A]$  that sends  $X : SA^{op} \rightarrow Set^A$  to the functor whose component at  $a$  is given as:

$$(\Sigma X)a = \coprod_{o \in O((k, \alpha, (n_i, \beta_i)_{1 \leq i \leq k}, \langle a, \vec{a}, \vec{b}_i \rangle))} \overline{\alpha}_{\widehat{SA}}(Xa_1(\overline{\beta}_1(\vec{b}_1, -)), \dots, Xa_k(\overline{\beta}_k(\vec{b}_k, -)))$$

Note that the fact that this formula describes the component at  $a$  corresponds to the fact that the sum is taken over operations for which the first type in its arity is also  $a$ .

The functor constructed in the proposition agrees with the functors Fiore et al and Tanaka generated from their signatures; and the category  $\Sigma\text{-Alg}$  of algebras for the functor agrees with their constructions too. Following Fiore et al, we overload notation by denoting both the signature and the functor it generates by  $\Sigma$ .

**Example 5.1.** Let  $S$  be  $T_{fp}$ , i.e., consider Fiore et al's cartesian binders. Our  $k$  is their  $k$ . If we let  $k$  consist of elements  $0, 1, \dots, k-1$ , our  $\alpha$  is the object  $0 \times 1 \times (k-1)$  of  $T_{fp}k$ , generating the functor

$$[\mathbb{F}, Set]^k \longrightarrow [\mathbb{F}, Set]$$

defining the  $k$ -fold product. Our  $n_i$  is their  $n_i$ . And our  $\beta_i$ , similarly to  $\alpha$ , is the object of  $T_{fp}(n_i + 1)$  generating the functor

$$\overline{\beta}_{i\mathbb{F}} : \mathbb{F}^{n_i+1} \longrightarrow \mathbb{F}$$

that sends  $(a_1, \dots, a_{n_i}, b)$  to  $a_1 + \dots + a_{n_i} + b$ . So every one of Fiore et al's binding signatures generates one of our binding signatures, and the endofunctor we define on  $[\mathbb{F}, Set]$  agrees, when restricted to one of their binding signatures, with their construction.

For a specific example, the untyped  $\lambda$ -calculus

$$M ::= x \mid \lambda x.M \mid \text{app}(M, M)$$

has two operators  $\lambda$  and  $\text{app}$ , with arities, in Fiore et al's terms, given by  $\langle 1 \rangle$ , and  $\langle 0, 0 \rangle$ , respectively. Let  $2$  be defined to have elements  $x$  and  $y$ . Then, in our terms, for the first operator  $\lambda$ , the arity is given by  $k = 1$ ,  $\alpha \in T_{fp}(1)$  is  $1$ ,  $n_1$  is  $1$ , and  $\beta_1$  is the element  $x \times y$  of  $T_{fp}(2)$ . And for the application  $\text{app}$ ,  $k = 2$ ,  $\alpha$  is the element  $x \times y$  of  $T_{fp}(2)$ ,  $n_0 = n_1 = 0$ , and both  $\beta_i$ 's are given by  $1$  seen as an element of  $T_{fp}(1)$ .

The corresponding binding signature for simply typed  $\lambda$ -calculus follows merely by decorating the untyped binding signature with types, according to the typed formulation of the signatures. Let  $T$  be the set of simple types generated from (some) given set of base types. Then the typed binding signature for simply typed lambda calculus is  $\Sigma_{\lambda \rightarrow} = (T, \mathcal{O}_{\lambda \rightarrow})$ , where  $\mathcal{O}_{\lambda \rightarrow}$  is a function such that, with the convention of taking  $x$  and  $y$  as objects of  $2$ , for any types  $\sigma, \tau \in T$ , the arity  $(1, 1, (1, x \times y), \langle \sigma \rightarrow \tau, \tau, \sigma \rangle)$  is sent to the set  $\{\lambda\}$ , and the arity  $(2, x \times y, ((0, 1), (0, 1)), \langle \tau, \sigma \rightarrow \tau, \sigma \rangle)$  to  $\text{app}$ .

Even in the untyped setting, not only are our binding signatures a priori more general than those of Fiore et al, but there seems to be one of our binding signatures for which there is none of Fiore et al's signatures with an equivalent category of algebras.

**Example 5.2.** Consider the signature in our sense consisting of one arity, with  $k = 1$ , with  $\alpha$  being the generating object  $1$  of  $T_{fp}1$ , and with  $\beta_1$  given by the pair  $y \times y$  in the notation of Example 5.1. An algebra would consist of a presheaf  $X$  together with a natural transformation

$$X(2 \times -) \longrightarrow X(-)$$

which does not appear to be constructable as an algebra for any signature in the sense of Fiore et al: note that  $y \times y$  generates  $X(2 \times -)$  rather than  $X(2 + -)$ .

The signature in Example 5.2 does not seem to have computational significance. That does not unduly perturb us: our main theorem about signatures is a positive one, asserting that any signature yields initial algebra semantics, so including uninteresting examples within that result does not bother us. Example 5.2 is sufficiently simple that it obviously extends to our other examples of pseudo-monads. In syntactic examples, one can put simple syntactic conditions on the choice of  $\beta_i$ 's along the lines of demanding precisely one occurrence of  $y$ , but we do not currently see a natural restriction at the level of generality of this paper that would restrict our definition so that it agrees in the case of  $S$  being  $T_{fp}$  with that of Fiore et al.

To enrich the work of this section, one could use exactly the definition of typed binding signature and the construction of an endofunctor as in the unenriched case subject to systematic generalisation of  $Cat$  to  $V\text{-}Cat$ , pseudo-monad to pseudo- $V\text{-}Cat$ -monad, and  $Set$  to  $V$ . That would allow for enriched versions of finite and linear contexts, the contexts of  $BI$ , and naming. But it is not the most general or natural extension to enrichment, and would not include structures such as finite cotensors or finite limits. To include those, one needs to add a list of  $1 + k$  elements of  $V$  to the definition of typed binding signature. The reason is that, in the enriched setting, even without binding, an algebraic operation is typically of the form

$$X^u \longrightarrow X^v$$

where  $v$  need not be a natural number [15]. For instance, if  $V$  is *Poset*, a possible  $v$  is Sierpinski space. So, unlike the unenriched setting, to give an operation as above need not be equivalent to giving several operations with codomain  $X$ , and so we need to add the parameter  $v$ . The arity  $u$  is already incorporated into the choice of object  $\alpha$  of  $Sk$ . The same holds for each  $n_i$ , hence our need for  $1 + k$  elements of  $V$ .

We still obtain an endo- $V$ -functor on the  $V$ -category  $[(SA)^{op}, V^A]$ , but its expression is a little more complex. To give an operation as above is equivalent to giving a map

$$v \otimes X^u \longrightarrow X$$

so one just needs systematically to add the  $1 + k$  elements of  $V$  determined by the arity of each operation to the expression. With that mild addition, the enriched theory is otherwise a routine generalisation of the unenriched theory.

## 6. Typed Initial Algebra Semantics

The central theorem of Fiore et al, albeit with a small error in the propositions leading up to it [6], then Tanaka [28], then us in the untyped unenriched setting [32], was an initial algebra semantics theorem. With care for detail, one can check that the proofs given in [32] extend to the typed and enriched setting. In fact, the proof of Proposition 6.1 given in [32] is unduly complicated and can be replaced by a simple use of the Yoneda lemma. But the proofs there are correct and generalise without fuss. So in this section, for simplicity, we state the series of results used in [32] in the typed setting, referring the reader to [32] for proofs that extend directly, while remarking that Proposition 6.1 is actually routine.

**Lemma 6.1.** Let  $S$  be a pseudo-monad on  $Cat$ , let  $\beta \in S(n+1)$ ,  $\vec{b} = b_1, \dots, b_n$ ,  $b_i, c \in SA$ , for  $1 \leq i \leq n$ , and let  $(C, h)$  be an  $S$ -algebra. Then

$$\overline{\beta_{SA}(\vec{b}, c)}_{C^-} = \overline{\beta}_C(\overline{b_1}_C(-), \dots, \overline{b_n}_C(-), \overline{c}_C(-)).$$

Observe also that when  $\beta = \iota(b_i) \in SA$ , it follows that  $\overline{\beta}_C = \pi_{b_i} : C^A \longrightarrow C$ .

**Proposition 6.1.** Each  $\beta \in S(n+1)$  induces a canonical natural transformation

$$(Xa)(\overline{\beta}_{SA}(\vec{b}, -)) \bullet_A Y \longrightarrow (Xa \bullet_A Y)(\overline{\beta}_{SA}(\vec{b}, -))$$

**Theorem 6.1.** There is a canonical strength of the endofunctor  $\Sigma$  over  $\bullet$

$$\Sigma X \bullet Y \longrightarrow \Sigma(X \bullet Y)$$

for pointed objects  $Y$ .

**Corollary 6.1.** For any binding signature  $\Sigma$ , if  $T_\Sigma$  is the free monad generated by  $\Sigma$  on the category  $[(SA)^{op}, Set^A]$ , it follows that  $T_\Sigma$  has a canonical strength over pointed objects with respect to  $\bullet$ .

**Corollary 6.2.** For any binding signature  $\Sigma$ , the object  $T_\Sigma(1)$  of  $[(SA)^{op}, Set^A]$  has a canonical monoid structure on it.

**Definition 6.1.** Let  $F$  be a strong (over pointed objects) endofunctor on a monoidal closed category  $(C, \cdot, I)$ . An  $F$ -monoid  $(X, \mu, \iota, h)$  consists of a monoid  $(X, \mu, \iota)$  in  $C$  and an  $F$ -algebra  $(X, h)$  such that the diagram

$$\begin{array}{ccc} F(X) \cdot X & \xrightarrow{t_{X,X}} & F(X \cdot X) \xrightarrow{F\mu} FX \\ \downarrow h \cdot X & & \downarrow h \\ X \cdot X & \xrightarrow{\mu} & X \end{array}$$

commutes.

$F$ -monoids form a category with maps given by maps in  $C$  that preserve both the  $F$ -algebra structure and the monoid structure.

**Theorem 6.2.** For any binding signature  $\Sigma$ , the object  $T_\Sigma(1)$  of the category  $[(SA)^{op}, Set^A]$  together with its canonical  $\Sigma$ -algebra structure and monoid structure, form the initial  $\Sigma$ -monoid.

That is a typed version of the central and final theorem of Tanaka's paper and of one of the two equivalent versions of the central and final result of Fiore et al's paper, exhibiting initial algebra semantics for a binding signature. The enriched version of this is entirely routine, subject to the refined definition of typed binding signature and the refined construction of  $\Sigma$  we described at the end of Section 5.

## 7. Further work

Although the notion of binder is syntactic, we have not given a general syntax in this paper. So an obvious question to address now is to provide syntax that corresponds to at least a class of the structures we have described here, enough to include at least the mixed variable binders of the Logic of Bunched Implication. It seems unlikely that there is a syntax to be found at the full generality of this paper, but there should be something interesting at a level of generality that is included in that of this paper and extends the leading classes of examples. The notion of a pseudo-commutative monad [9] may be relevant.

Other development of binders has involved the use of sheaves rather than presheaves, e.g., in [7]. Sheaves appear for example if one wants to justify decidable equality of variables, which presheaves do not support. The focus on sheaves may be misleading: consistent with the algebraic techniques of this paper, preservation of classes of limits, of which the sheaf condition is an instance, may be more fundamental. So, in due course, we plan to refine the pseudo-monadic approach of this paper to allow for limit preservation conditions: such categories are given by free cocompletions [13, 14], which seems to be important.

Hofmann [8] studied logical principles on binding structures. Accordingly, we too should like to refine our approach to incorporate logical principles such as induction over higher-order terms.

Ideally, we should also like to give an account of typed binding signatures that includes structure on the types. But we do not see any promising avenue yet. Product, coproduct, and exponential are all binary operators on types, but have three very different sorts of behaviour in regard to terms, and we



cannot currently see any reasonable way in which to deal with that. But it would still be a good question to resolve.

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## A. Enriched 2-categories

Over recent years, there has been a growing body of work on very general notions of monoidal 2-category or bicategory, along the lines of [23]. What we need in this paper is considerably less complicated than that, but it is convenient to use some of the same terminology. So the terminology we use here is not standard in the literature on weak higher-dimensional categories.

The leading example we shall develop is given by  $V\text{-Cat}$  for an ordinary symmetric monoidal closed category  $V$ : if  $V$  is complete, cocomplete, and locally small,  $V\text{-Cat}$  is both a symmetric monoidal closed category and a 2-category [13], and in this paper, we need to deal with both structures and the relationship between them.

The key point for us is in modelling substitution: as we have shown before [30], for any pseudo-monad  $T$  on  $Cat$ , the category  $T1$  acquires a canonical monoidal structure that plays the role of substitution. But if we generalise from  $Cat$  to  $V\text{-Cat}$ , the existence of a pseudo-monad on  $V\text{-Cat}$  qua 2-category does not yield the substitution structure we seek.

On the other hand,  $V\text{-Cat}$  has a canonical symmetric monoidal closed structure on it [13], and if we had a  $V\text{-Cat}$ -enriched monad  $T$  on  $V\text{-Cat}$ , we would obtain a substitution monoidal structure on  $TT$

where  $\mathcal{I}$  is the unit  $V$ -category. But in the examples, we do not have a  $V$ -Cat-enriched monad on  $V$ -Cat for precisely the same reasons as, in the unenriched setting, we do not have a 2-monad on  $Cat$  but rather a pseudo-monad.

So we need to combine the two ideas: we need to account for the symmetric monoidal closed structure of  $V$ -Cat while also allowing for non-strictness, thus our need to develop notions such as that of pseudo- $W$ -monad for a symmetric monoidal closed 2-category  $W$ , with leading example given by taking  $W$  to be  $V$ -Cat for a complete and cocomplete symmetric monoidal closed category  $V$ . So, in this appendix, we introduce the small amount of theory for categories enriched in symmetric monoidal closed 2-categories we need in order to support the semantics for variable binding we develop in the heart of the paper.

**Definition A.1.** A *monoidal 2-category* is a 2-category  $W$  together with a 2-functor  $\otimes : W \times W \longrightarrow W$ , an object  $I$ , and invertible 2-natural transformations with components

$$a_{XYZ} : (X \otimes Y) \otimes Z \longrightarrow X \otimes (Y \otimes Z)$$

$$l_X : I \otimes X \longrightarrow X$$

and

$$r_X : X \otimes I \longrightarrow X$$

subject to the two usual equations in the definition of a monoidal category [13] seen as equations between 2-natural transformations. A monoidal 2-category is *closed* if, for every object  $X$ , the 2-functor  $- \otimes X : W \longrightarrow W$  has a right 2-adjoint.

**Definition A.2.** A *symmetry* on a monoidal 2-category  $W$  is an invertible 2-natural transformation with components

$$c_{XY} : X \otimes Y \longrightarrow Y \otimes X$$

subject to the usual equations in the definition of a symmetry for a monoidal category [13] seen as equations between 2-natural transformations. A *symmetric monoidal 2-category* is a monoidal 2-category equipped with a symmetry.

**Example A.1.** Let  $V$  be a complete and cocomplete symmetric monoidal closed category. It is shown in [13] that  $V$ -Cat supports both a symmetric monoidal closed structure and the structure of a 2-category. It is also shown there that the two structures together satisfy the axioms for a symmetric monoidal closed 2-category. Our leading example of such a  $V$  is given by  $\omega Cpo$  with its cartesian closed structure. When the monoidal structure on  $V$  is cartesian structure, it follows that the induced monoidal structure on  $V$ -Cat is also cartesian [13]. That makes for simpler calculation at some points.

**Definition A.3.** Given a symmetric monoidal closed 2-category  $W$ , a  $W$ -2-category  $\mathbb{C}$  is a  $W$ -category using the standard definition of enriched category theory for a symmetric monoidal closed category  $W$  [13], i.e., a set  $Ob\mathbb{C}$ , homobjects  $\mathbb{C}(X, Y)$  of  $W$ , a composition law and identity elements, subject to associativity and unit laws.

**Example A.2.** If  $W$  is a symmetric monoidal closed 2-category,  $W$  itself canonically supports the structure of a  $W$ -category for precisely the same reasons as hold in ordinary enriched category theory. In Sections 2 and 3, we develop further classes of examples generated primarily by  $W$ -2-categories of pseudo- $W$ -algebras of pseudo- $W$ -monads on  $W$ .

**Definition A.4.** Given a symmetric monoidal closed 2-category  $W$ , a  $W$ -2-functor is a  $W$ -functor as in [13], and a  $W$ -2-natural transformation is a  $W$ -natural transformation also as in [13].

These definitions mean that we could equally use the terminology  $W$ -category in place of  $W$ -2-category, etcetera. But we prefer the latter here as we soon need to generalise from  $W$ -functors to pseudo-versions of the notion, similarly for  $W$ -natural transformations, and that only makes sense in a two-dimensional setting. So for us to use the prefix 2 reminds us of our mathematical setting.

It follows immediately from the situation for ordinary enriched category theory that  $W$ -2-categories,  $W$ -2-functors and  $W$ -2-natural transformations form a 2-category.

**Proposition A.1.** Every  $W$ -2-category  $\mathbb{C}$  has a canonical underlying 2-category  $\mathbb{C}_0$  generated by the representable 2-functor

$$W(I, -) : W \longrightarrow Cat$$

**Proof:**

$Ob\mathbb{C}_0$  is defined to be  $ObC$ . For each pair of objects  $X$  and  $Y$  of  $\mathbb{C}$ , the homcategory  $\mathbb{C}_0(X, Y)$  is defined to be  $W(I, \mathbb{C}(X, Y))$ . Composition in  $\mathbb{C}_0$  is induced by composition in  $\mathbb{C}$  together with the fact that  $I$  is the unit of  $W$ . It is routine to verify the associativity and unit axioms.  $\square$

The proposition allows us sensibly to speak of 1-cells and 2-cells in a  $W$ -category  $\mathbb{C}$ : just as for ordinary enriched categories, a 1-cell in  $\mathbb{C}$  is defined to be a 1-cell in the 2-category  $\mathbb{C}_0$  [13], and, extending the situation for ordinary enriched category theory, a 2-cell in  $\mathbb{C}$  is defined to be a 2-cell in the 2-category  $\mathbb{C}_0$ . We can now define a sensible notion of  $W$ -modification as follows:

**Definition A.5.** A  $W$ -modification

$$\tau : \alpha \rightarrow \beta : H \Rightarrow K$$

is an assignment to each object  $X$  of  $\mathbb{C}$ , a 2-cell in  $\mathbb{D}$

$$\tau_X : \alpha_X \Rightarrow \beta_X : HX \longrightarrow KX$$

in  $\mathbb{D}$  subject to the equality of 2-cells in  $W$  as follows:

$$\mathbb{D}(\tau_X, KY) \cdot K_{XY} = \mathbb{D}(HX, \tau_Y) \cdot H_{XY}$$

This definition, with the canonically induced composition, extends the situation of ordinary enriched category theory to make  $W$ -Cat into a 3-category. Moreover, the construction of the proposition extends routinely from  $W$ -categories to  $W$ -functors,  $W$ -natural transformations and  $W$ -modifications. Thus we can again extend the situation of ordinary enriched category theory to obtain the following:

**Theorem A.1.** The representable 2-functor

$$W(I, -) : W \longrightarrow \mathcal{C}at$$

generates a canonical 3-functor

$$W - \mathcal{C}at \longrightarrow 2 - \mathcal{C}at$$

All we have written so far could have been more generally expressed in terms of a pair of symmetric monoidal closed categories  $V$  and  $W$  and a symmetric monoidal closed adjunction between them, cf [4]: one takes  $W$  as we have done and takes  $V$  to be  $\mathcal{C}at$ . But now we need pseudo-versions of the various definitions, and for that, we need two-dimensional structure, so cannot generalise from  $\mathcal{C}at$  to arbitrary  $V$ .

**Definition A.6.** A *pseudo- $W$ -functor*  $H : \mathbb{C} \longrightarrow \mathbb{D}$  sends each object  $X$  of  $\mathbb{C}$  to an object  $HX$  of  $\mathbb{D}$ , has a map in  $W$  from  $\mathbb{C}(X, Y)$  to  $\mathbb{D}(HX, HY)$ , and preserves composition and identities up to coherent isomorphism in  $W$ , the coherence conditions being identical to those for pseudo-functors listed in [30, 29].

**Definition A.7.** A *pseudo- $W$ -natural transformation*  $\alpha : H \Rightarrow K$  between pseudo-functors assigns, to each object  $X$  of  $\mathbb{C}$ , a 1-cell  $\alpha_X : HX \longrightarrow KX$  that is  $W$ -natural up to coherent isomorphism in  $W$ , with coherence conditions those for pseudo-natural transformations as in [30, 29] but read as being enriched in  $W$ .

The definition of  $W$ -modification above extends routinely from  $W$ -natural transformations to pseudo- $W$ -natural transformations. It follows immediately from the above that every pseudo- $W$ -functor has an underlying pseudo-functor, and similarly for pseudo- $W$ -natural transformations and  $W$ -modifications.

These definitions yield a definition of pseudo- $W$ -monad as follows:

**Definition A.8.** A *pseudo- $W$ -monad* on a  $W$ -2-category  $\mathbb{C}$  consists of

- a pseudo- $W$ -functor  $T : \mathbb{C} \longrightarrow \mathbb{C}$
- a pseudo- $W$ -natural transformation  $\mu : T^2 \longrightarrow T$
- a pseudo- $W$ -natural transformation  $\eta : Id \longrightarrow T$
- invertible  $W$ -modifications  $\tau, \lambda$  and  $\rho$  for associativity and left and right unit respectively

subject to the two coherence axioms in the definition of pseudo-monad listed in [30, 29]

A fortiori from the situation for pseudo- $W$ -functors, etcetera, every pseudo- $W$ -monad has an underlying pseudo-monad.

The definition of pseudo- $W$ -algebra for a pseudo- $W$ -monad  $T$  is identical to that of a pseudo-algebra for the underlying pseudo-monad  $T_0$  of  $T$ . One can routinely and canonically enrich the usual notions of pseudo-map and algebra 2-cell to define a  $W$ -2-category  $Ps\text{-}T\text{-}Alg$  for a pseudo- $W$ -monad on a  $W$ -2-category  $\mathbb{C}$ , with a canonical forgetful  $W$ -2-functor from  $Ps\text{-}T\text{-}Alg$  to  $\mathbb{C}$ .

We finally need the notion of a  $W$ -bicategory, but again, the generalisation from the standard notion of bicategory is routine: one has a set of objects, for each pair of objects, a hom-object of  $W$ , together with composition and unit data that satisfy the usual axioms for a bicategory but enriched in  $W$ .